

ICTM State Contest 2009
Voting Theory Questions

1. The AP Top 25 College Men's Basketball poll is a ranking system that assigns total points based on 25 points for a first-place vote, 24 points for a second-place vote, and so on down through one point for a 25th place vote. The voters' points are summed to determine a team's ranking for the week.

In the AP ranking based on team records through February 15, Oklahoma, with four first place votes and 1709 total points, was ranked second among the top 25 teams.

- a. If the total number of points to be distributed was 23,400, how many voters were there?
- b. If no voters ranked Oklahoma lower than 3rd place, how many second place votes and how many third place votes did Oklahoma receive for the week ending February 15?

2. Voters A, B, C, D, and E use the weighted voting system [8: 5, 3, 1, 1, 1].

- a. What are the minimal winning coalitions in this system?
- b. Does any voter in this system have veto power? If so, who?
- c. Calculate the Banzhaf index for this voting system.
- d. Under the Banzhaf model, what percent of the voting power does voter B have?

3. Consider again the weighted voting system [8: 5, 3, 1, 1, 1].

- a. How many permutations of the voters are there?
- b. Under the Shapley-Shubik model, in how many permutations will B be the pivotal voter?
- c. What is the Shapley-Shubik index for voter B?

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Voting Theory Solutions

1. a. Each voter has $25 + 24 + \dots + 2 + 1 = \frac{25 \cdot 26}{2} = 325$ points to assign. The number of voters is thus $\frac{23,400}{325} = 72$.

b. Since there are 72 voters, and 4 of them ranked Oklahoma in first place, the second and third place votes must total 68. Let x be the number of second place votes. Then the number of third place votes is $68 - x$. An expression for the number of points Oklahoma received is thus $25 \cdot 4 + 24 \cdot x + 23 \cdot (68 - x)$. We solve the equation

$$\begin{aligned} 25 \cdot 4 + 24 \cdot x + 23 \cdot (68 - x) &= 1709 \\ 100 + 24 \cdot x + 1564 - 23 \cdot x &= 1709 \\ x &= 45 \end{aligned}$$

$$68 - x = 68 - 45 = 23$$

Oklahoma received 45 second place votes and 23 third place votes.

2. a. The minimal winning coalitions are $\{A, B\}$ and $\{A, C, D, E\}$.

b. Yes, A has veto power. No issue can pass without A's vote.

c.

<u>winning coalition</u>	<u>weight</u>	<u>extra votes</u>	<u>critical voters</u>				
			A	B	C	D	E
{A, B}	8	0	1	1	0	0	0
{A, B, C}	9	1	1	1	0	0	0
{A, B, D}	9	1	1	1	0	0	0
{A, B, E}	9	1	1	1	0	0	0
{A, B, C, D}	10	2	1	1	0	0	0
{A, B, C, E}	10	2	1	1	0	0	0
{A, B, D, E}	10	2	1	1	0	0	0
{A, C, D, E}	8	0	1	0	1	1	1
{A, B, C, D, E}	11	3	<u>1</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
			9	7	1	1	1

Each voter is a critical voter in the same number of blocking coalitions as winning coalitions. Thus, the Banzhaf index for this system is (18, 14, 2, 2, 2).

d. B's share of the voting power is $\frac{14}{18+14+2+2+2} = \frac{14}{38} = \frac{7}{19} \approx 37\%$.

3. a. There are $5! = 120$ permutations of the five voters.

b. B has only 3 votes. Thus, B will never be a pivotal voter if he is the first voter in a permutation. Also, the other four voters have a combined total of 8 votes. Thus, B will never be a pivotal voter if he is the last member of a permutation. We consider the cases where B is the 2nd, 3rd, and 4th member of the permutation.

case 1: B is the second member of the permutation B

B is pivotal only when A is the first member of the permutation; since the remaining one-vote members can be arranged in $3! = 6$ ways, there are 6 permutations in which B is pivotal

case 2: B is the third member of the permutation B

B will be pivotal only when A is the first or second member of the permutation; in each case there will be 6 ways to arrange the other one-vote members; thus, B will be pivotal in twelve of these permutations

case 3: B is the fourth member of the permutation B

Again, B will be pivotal only when A is the first, second, or third member of the permutation; in each of these cases there will be 6 permutations of the other members; thus, B will be pivotal in 18 of these permutations

The total number of permutations in which B is pivotal is thus $6 + 12 + 18 = 36$.

c. The Shapley-Shubik index for voter B is $\frac{36}{120} = \frac{3}{10}$. In the Shapley-Shubik model, B has 30% of the voting power.

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Voting Theory Extemporaneous Questions

Voters A, B, and C want to use the weighted voting system $[q: 8, 4, 1]$.

1. What are the possible values they can use for q ?
2. Which values of q will result in exactly one voter with veto power? Who? Why?
3. Which values of q will result in one or more dummies? Who? Why?

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Voting Theory Extemporaneous Solutions

1. Since we require that a quota be greater than half the sum of the weights, in this system we require $q > \frac{8+4+1}{2} = \frac{13}{2}$. Possible values for q are thus 7, 8, 9, 10, 11, 12, 13.
2. A voter has veto power if all coalitions consisting of only other voters are losing coalitions. Using values of 7, 8, or 9 for q would ensure that voter A and only voter A has veto power. (Using a quota of 10, 11, or 12 would give both voter A and voter B veto power. Using a quota of 13 would give all three voters veto power.)
3. A dummy is a voter that is not a critical voter in any winning coalition. Using values of 7 or 8 for q would make both voter B and voter C dummies. Using values of 10, 11, or 12 for q would make voter C a dummy. (Letting $q = 9$ or $q = 13$ yields a voting system with no dummies.)